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The algebraic structures of isospectral Lax operators and applications to integrable equations

Wen-xiu Ma

CCAST (World Laboratory) PO Box 8730, Beijing 100080, People's Republic of China and (mailing address)
Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

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Abstract. For a general spectral operator, we establish types of algebraic structures of the spaces of the corresponding isospectral Lax operators, which essentially form the theoretical basis of the Lax operator method. Furthermore we introduce the concepts of τ-algebras and master algebras to describe time-polynomial-dependent symmetries of nonlinear integrable equations. Finally we apply our theory of Lax operators to the KP hierarchy of integrable equations as an illustrative example, and thus obtain the master symmetry algebra of the KP hierarchy.

1. Introduction

It is well known that many nonlinear integrable equations in $1+1$ dimensions possess some common aspects: Lax representations, infinitely many symmetries and conserved quantities, existence of bi-Hamiltonian formulations, etc (see Newell 1985, Olver 1986, and Magri 1980). A recursion operator with the hereditary property plays a central role in investigating the above algebraic properties. Recently the above theory has been extended to integrable equations in $2+1$ dimensions (see Santini and Fokas 1988, Fokas and Santini 1988a). In particular, Santini and Fokas have found the multidimensional analogue of the recursion operator called the extended recursion operator for several integrable equations in $2+1$ dimensions (see Santini 1989, Fokas and Santini 1988b). However, for a given equation in $2+1$ dimensions, it not very easy to construct an extended recursion operator which can admit the bi-Hamiltonian factorization. To avoid this difficulty, Cheng *et al* (1988) and Cheng (1988, 1991) have proposed a direct method of Lax operators by discussing the algebraic properties for special integrable equations, based upon the ideal of the master-symmetry method of Fuchssteiner (1983), Fokas and Fuchssteiner (1981), Oevel and Fuchssteiner (1982), Chen *et al* (1985, 1982, 1983), Chen and Lin (1987). We shall generalize the Lax operator method to integrable equations associated with a rather general spectral operator, by exposing a property of Gateaux derivative operators of matrix differential operators. The theory of this paper is applicable to integrable equations both in $1+1$ and in $2+1$ dimensions.

This paper is organized as follows. In section 2, for any matrix differential spectral operator, we give types of product operators of isospectral Lax operators which correspond to the commutator of vector fields and display the Lie algebraic structure of a quotient algebra of the Lax operator algebra. Section 3 discusses the relations between symmetries of integrable equations and subalgebras of the Lax operator algebra. The introduced τ-algebras and master algebras play an analogous role to recursion operators. In section 4, as an application of our theory, we present a τ-algebra and a master algebra for the KP hierarchy of integrable equations and thus derive the explicit formulas for K-symmetries and every-degree master symmetries of the KP hierarchy of integrable equations.

In the following, we give the fundamental notation (see also Ma 1992).

1.1. The independent and dependent variables

Let the independent variables $x \in R^p$, $t \in R$, and the dependent variables $u^i = u^i(x,t)$, $1 \leqslant i \leqslant q$, belong to Schwartz space over R^p for any fixed $t \in R$. We denote by $S^q(R^p, R)$ all vectors $u = (u^1, \dots, u^q)^T$ of dependent variables of that kind.

1.2. The spaces $\mathcal{B}^r, \mathcal{V}^r$ and $\mathcal{V}_0^r$

For $\alpha = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \in Z$ and $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant p$, we write $D^\alpha = (\mathrm{d}/\mathrm{d}x^1)^{\alpha_1} \cdots (\mathrm{d}/\mathrm{d}x^p)^{\alpha_p}$, $|\alpha| = \sum_{i=1}^p \alpha_i$. Let $\mathcal{B}$ denote all complex (or real) functions $P[u] = P(x,t,u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x)$ (as functions of x), and set $\mathcal{B}^r = \{(P_1, \dots, P_r)^T \mid P_i \in \mathcal{B},\ 1 \leqslant i \leqslant r\}$. Let $\mathcal{V}^r$ denote all linear operators $\Phi = \Phi(x,t,u) : \mathcal{B}^r \to \mathcal{B}^r$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x)$, and by $\mathcal{V}_0^r (\sqsubseteq \mathcal{V}^r)$ all matrix differential operators $\Phi : \mathcal{B}^r \to \mathcal{B}^r$ with the special form

$$\Phi = (\Phi_{ij})_{r\times r} \qquad \Phi_{ij} = \sum_{|\alpha| \leqslant \alpha(i,j)} P_\alpha^{ij}[u] D^\alpha \qquad P_\alpha^{ij}[u] \in \mathcal{B}\,. \tag{1.1}$$

Note that here the space $\mathcal{B}$ includes non-local functions, for example, the Hilbert transform of u, $Hu = \frac{1}{\pi} P \int_{-\infty}^{\infty} u(y,t)/(y-x)\,\mathrm{d}y$. Therefore it is an extension of the space $\mathcal{A}$, which consists of local functions only (see Ma 1991b).

1.3. The Gateaux derivative

For a vector function $K \in \mathcal{B}^r$, define its Gateaux derivative in the direction $S \in \mathcal{B}^q$ as

$$K'[S] = \frac{\partial}{\partial \epsilon} K(u + \epsilon S)|_{\epsilon=0} \tag{1.2}$$

and for two vector fields $K, S \in \mathcal{B}^q$, define the product vector field as

$$[K, S] = K'[S] - S'[K] \tag{1.3}$$

which has been shown to be a commutator operation of $\mathcal{B}^q$ by Bowman (1987). For an operator $\Phi \in \mathcal{V}^r$, define its Gateaux derivative operator $\Phi' : \mathcal{B}^q \to \mathcal{V}^r$ as

$$\Phi'[K]S = \frac{\partial}{\partial \epsilon} \Phi(u + \epsilon K) S\,|_{\epsilon=0} \qquad K \in \mathcal{B}^q \quad S \in \mathcal{B}^r\,. \tag{1.4}$$

Throughout this paper, we always choose the spectral operator $L = L(x,u) : \mathcal{B}^r \to \mathcal{B}^r$ to be a matrix differential operator with the form (1.1) and assume that its Gateaux derivative operator $L' : \mathcal{B}^q \to \mathcal{V}_0^r$ is an injective linear homomorphism.

2. Lax operators and their algebraic structures

We suppose that the equations $u_t = X$, $u_t = Y$ ($X, Y \in \mathcal{B}^q$) possess the Lax representations $L_t = [A, L]$, $L_t = [B, L]$ ($A, B \in \mathcal{V}^r$), respectively. In the Lax theory, there is the basic question: does the equation $u_t = [X, Y]$ possess the Lax representation, too? If the answer is 'yes', what form does the operator $C \in \mathcal{V}^r$ of its Lax representation $L_t = [C, L]$ possess?

In this section, for the spectral operator $L : \mathcal{B}^r \to \mathcal{B}^r$, we establish a kind of algebraic structure of the space of the corresponding isospectral Lax operators and further derive the Lie algebraic structure of a quotient algebra of the Lax operator algebra. Therefore we give a complete answer to the above basic question.

Definition 2.1. Let $A \in \mathcal{V}^r$. If there exists a vector field $X \in \mathcal{B}^q$ such that $[A, L] = L'[X]$, then A is called an isospectral Lax operator, or a Lax operator for short, and X called an eigenvector field of the Lax operator A. Moreover, we denote by $\mathcal{M}_\infty$ all Lax operators and by $E(\mathcal{N}_\infty)$, all eigenvector fields of Lax operators in a subset $\mathcal{N}_\infty$ of $\mathcal{M}_\infty$.

Note that when $[A, L] = L'[X]$, $A \in \mathcal{V}^r, X \in \mathcal{B}^q$, the equation $u_t = X$ possesses the Lax representation $L_t = [A, L]$ by $L_t = L'[u_t]$. Moreover a Lax operator only has an eigenvector field as L' is injective. Therefore we can further give the following definition.

Definition 2.2. Let two Lax operators $A, B \in \mathcal{M}_\infty$ have eigenvector fields $X, Y \in E(\mathcal{M}_\infty)$, respectively. Then we define the product operator of two Lax operators A, B as follows:

$$[\![A, B]\!] = A'[Y] - B'[X] + [A, B]. \tag{2.1}$$

We shall show that this product operator $[\![A, B]\!]$ just corresponds to the commutator $[X, Y]$. To this end, we first need the following basic result.

Theorem 2.1. Let $P = P(x, t, u) \in \mathcal{B}$, $K = K(x, t, u), S = S(x, t, u) \in \mathcal{B}^q$; then we have the relation

$$(P'[K])'[S] - (P'[S])'[K] = P'[T] \qquad T = [K, S]. \tag{2.2}$$

Proof. By the definition of the Gateaux derivative, we have

$$\begin{aligned}(P'[K])'[S] &= \left(\frac{\partial}{\partial\epsilon}P(u+\epsilon K)|_{\epsilon=0}\right)'[S] = \frac{\partial^2}{\partial\delta\partial\epsilon}P(u+\delta S+\epsilon K(u+\delta S))|_{\delta,\epsilon=0}\\ &= \frac{\partial^2}{\partial\delta\partial\epsilon}P(u+\delta S+\epsilon K+\epsilon\delta K'[S])|_{\delta,\epsilon=0}\\ &= \frac{\partial^2}{\partial\delta\partial\epsilon}P(u+\delta S+\epsilon K)|_{\delta,\epsilon=0} + \frac{\partial}{\partial\mu}P(u+\mu K'[S])|_{\mu=0}\,.\end{aligned}$$

At the same time, we similarly have

$$(P'[S])'[K] = \frac{\partial^2}{\partial\delta\partial\epsilon}P(u+\delta K+\epsilon S)|_{\delta,\epsilon b=0} + \frac{\partial}{\partial\mu}P(u+\mu S'[K])|_{\mu=0}\,.$$

Thus we obtain

$$(P'[K])'[S] - (P'[S])'[K] = \frac{\partial}{\partial \mu}[P(u+\mu K'[S]) - P(u+\mu S'[K])]\,|_{\mu=0}$$
$$= P'[K'[S]] - P'[S'[K]] = P'[T]$$

which completes the proof. □

From the above theorem, we can easily deduce the following corollary.

Corollary 2.1. Let $\Phi \in V_0^r$, $K, S \in B^q$. Then we have

$$(\Phi'[K])'[S] - (\Phi'[S])'[K] = \Phi'[T] \qquad T = [K, S].$$

Theorem 2.2. Suppose that two Lax operators $A, B \in \mathcal{M}_\infty$ have eigenvector fields $X, Y \in E(\mathcal{M}_\infty)$, respectively. Then we have the equality

$$[[\![A, B]\!], L] = L'[Z] \qquad Z = [X, Y] \tag{2.3}$$

which shows that the product operator $[\![A, B]\!] \in V^r$ is a Lax operator, too, and that its eigenvector field is the commutator $[X, Y]$.

Proof. Since $\langle V^r, [\cdot,\cdot]\rangle$ is an operator Lie algebra, we have

$$[[A, B], L] = [A, [B, L]] - [B, [A, L]] = [A, L'[Y]] - [B, L'[X]].$$

Therefore,

$$[[\![A, B]\!], L] = [A'[Y], L] + [A, L'[Y]] - [B'[X], L] - [B, L'[X]]$$
$$= [A, L]'[Y] - [B, L]'[X]$$
$$= (L'[X])'[Y] - (L'[Y])'[X] = L'[Z] \qquad \text{(by corollary 2.1)}.$$

The rest is obvious and thus the result is proved. □

Evidently we see by (2.1) that the multiplication operation $[\![\cdot,\cdot]\!]$ is bi-linear and anti-commutative. Therefore noticing that the multiplication operation $[\cdot,\cdot]$ given by (1.3) satisfies Jacobi identity (see Bowman 1987), we obtain at once the following three results by the above theorem.

Corollary 2.2. $\langle \mathcal{M}_\infty, [\![\cdot,\cdot]\!]\rangle$ is an anti-commutative algebra and $\langle E(\mathcal{M}_\infty), [\cdot,\cdot]\rangle$ is a Lie algebra.

Corollary 2.3. Let $A, B, C \in \mathcal{M}_\infty$. Then we have

$$[[\![[\![A, B]\!], C]\!] + \text{cycle}(A, B, C), L] = 0. \tag{2.4}$$

Corollary 2.4. If $\mathcal{N}_\infty$ is a subalgebra of $\mathcal{M}_\infty$, then $E(\mathcal{N}_\infty)$ is a Lie subalgebra of $E(\mathcal{M}_\infty)$ and possesses the same algebraic structure as $\mathcal{N}_\infty$. Thus the space of the flows of the equations $u_t = X$, $X \in E(\mathcal{N}_\infty)$, possesses the same algebraic structure as $\mathcal{N}_\infty$, too.

Set $K(L) = \{A \in \mathcal{V}^r | [A, L] = 0\}$. Obviously $K(L)$ is a Lie subalgebra of $\langle \mathcal{V}^r, [\cdot,\cdot] \rangle$ and is also a subalgebra of $\langle \mathcal{M}_\infty, [\![\cdot,\cdot]\!] \rangle$. Thus $K(L)$ generates an equivalence relation $\sim$ of $\mathcal{V}^r$:

$$A \sim B \Longleftrightarrow [A, L] = [B, L] \qquad A, B \in \mathcal{V}^r .$$

We denote by $CL(A)$ the equivalence class to which $A \in \mathcal{V}^r$ belongs.

Proposition 2.1. $\langle K(L), [\![\cdot,\cdot]\!] \rangle$ is an ideal subalgebra of $\langle \mathcal{M}_\infty, [\![\cdot,\cdot]\!] \rangle$.

Proof. Let $A \in K(L)$, $B \in \mathcal{M}_\infty$. If follows from theorem 2.2 that $[\![A, B]\!], [\![B, A]\!] \in K(L)$, which implies that the result of the theorem is true. □

Based on the above proposition, we can generate a quotient algebra $CL(\mathcal{M}_\infty) = \mathcal{M}_\infty / K(L) = \{CL(A) | A \in \mathcal{M}_\infty\}$, whose multiplication operation is as follows:

$$[\![CL(A), CL(B)]\!] = CL([\![A, B]\!]) \qquad A, B \in \mathcal{M}_\infty . \tag{2.5}$$

To explain that the equality (2.5) makes sense, we may directly prove the following result. If $A_1 \sim A_2$, $B_1 \sim B_2, A_i, B_i \in \mathcal{M}_\infty$, $i = 1, 2$, then $[\![A_1, B_1]\!] \sim [\![A_2, B_2]\!]$. In the following we would like to show this. Suppose that $[A_i, L] = L'[X]$, $[B_i, L] = L'[Y]$, $i = 1, 2$. Then we have

$$\begin{aligned} [(A_1 - A_2)'[Y] - (B_1 - B_2)'[X], L] &= -[A_1 - A_2, L'[Y]] + [B_1 - B_2, L'[X]] \\ &= -[A_1, [B_1, L]] + [A_2, [B_2, L]] + [B_1, [A_1, L]] - [B_2, [A_2, L]] \\ &= -[[A_1, B_1], L] + [[A_2, B_2], L] . \end{aligned}$$

It follows that

$$[\![A_1, B_1]\!] \sim [\![A_2, B_2]\!]$$

which is just the desired result.

Theorem 2.3. The quotient algebra $\langle CL(\mathcal{M}_\infty), [\![\cdot,\cdot]\!] \rangle = \langle \mathcal{M}_\infty / K(L), [\![\cdot,\cdot]\!] \rangle$ is a Lie algebra and isomorphic to the Lie algebra $\langle E(\mathcal{M}_\infty), [\cdot,\cdot] \rangle$. Moreover the following mapping

$$\rho : CL(\mathcal{M}_\infty) \to E(\mathcal{M}_\infty) \qquad CL(A) \mapsto X \ ([A, L] = L'[X])$$

is a Lie algebraic isomorphism between Lie algebras $\langle CL(\mathcal{M}_\infty), [\![\cdot,\cdot]\!] \rangle$ and $\langle E(\mathcal{M}_\infty), [\cdot,\cdot] \rangle$.

Proof. Obviously, ρ is a linear isomorphism. If Lax operators $A, B \in \mathcal{M}_\infty$ have the eigenvector fields $X, Y \in E(\mathcal{M}_\infty)$, respectively, then we have

$$\rho([\![CL(A), CL(B)]\!]) = \rho(CL([\![A, B]\!])) = [X, Y] = [\rho(CL(A)), \rho(CL(B))] .$$

Thus by corollary 2.2, we obtain the result that $\langle CL(\mathcal{M}_\infty), [\![\cdot,\cdot]\!] \rangle$ is a Lie algebra and further we see that ρ is a Lie algebraic isomorphism. The proof is completed. □

Corollary 2.5. If an eigenvector field $X \in E(\mathcal{M}_\infty)$ corresponds to a Lax operator $A \in \mathcal{M}_\infty$, then the equivalence class $CL(A)$ is just all Lax operators to which the vector field X corresponds.

By now, we have systematically answered to the question posed at the beginning of this section. Corollary 2.5 also gives an answer to the second basic question in Marvulle and Wreszinski (1989): if an equation $u_t = X \, (X \in \mathcal{B}^q)$ has a Lax representation $L_t = [A, L] \, (A \in \mathcal{V}^r)$, how many different Lax operators A are associated with the same equation $u_t = X$?

3. Subalgebras and symmetries

From now on, for a subset V of some linear space (for instance, the space of Lax operators $\mathcal{M}_\infty$ or the space of vector fields $\mathcal{B}^q$), we always use $\mathrm{span}(V)$ to denote the subspace spanned by V.

Theorem 3.1. Let $\mathcal{N}_\infty$ be a subalgebra of $\mathcal{M}_\infty$. If Lax operators of $\mathcal{N}_\infty$ are not equivalent to each other, then $\langle \mathcal{N}_\infty, [\![\cdot,\cdot]\!]\rangle$ forms a Lie algebra and is isomorphic to the Lie algebra $\langle CL(\mathcal{N}_\infty), [\![\cdot,\cdot]\!]\rangle$ (or $\langle E(\mathcal{N}_\infty), [\cdot,\cdot]\rangle$).

Proof. By the hypothesis and corollary 2.3, we see that $\langle \mathcal{N}_\infty, [\![\cdot,\cdot]\!]\rangle$ forms a Lie algebra. We make the mapping

$$\rho : \mathcal{N}_\infty \to CL(\mathcal{N}_\infty) \qquad A \mapsto CL(A).$$

Evidently, ρ is linear and surjective. Moreover by the hypothesis, ρ is injective. Hence ρ is a linear isomorphism. In addition,

$$\rho([\![A,B]\!]) = CL([\![A,B]\!]) = [\![CL(A), CL(B)]\!] = [\![\rho(A), \rho(B)]\!] .$$

Therefore ρ is a Lie algebraic isomorphism. Now the proof is complete. □

This theorem gives an approach for proving that some set of Lax operators $\mathcal{N}_\infty$ ($\sqsubseteq \mathcal{M}_\infty$) is a Lie algebra with the multiplication operation $[\![\cdot,\cdot]\!]$. If we can verify that (1) $\mathcal{N}_\infty$ is closed under the multiplication operation $[\![\cdot,\cdot]\!]$ defined by (2.1), (2) Lax operators of $\mathcal{N}_\infty$ are not equivalent to each other, that is to say that the operator equation $[A, L] = 0$ with respect to A has the unique zero solution $A = 0$ in $\mathcal{N}_\infty$, or that Lax operators $A \in \mathcal{N}_\infty$ correspond one-to-one to eigenvector fields $X \in E(\mathcal{N}_\infty)$, then $\mathcal{N}_\infty$ constitutes a Lie subalgebra of $\mathcal{M}_\infty$ with the multiplication operation $[\![\cdot,\cdot]\!]$. In general, for a hierarchy of isospectral integrable equations $u_t = X_m = \Phi^m f_0 (\Phi \in \mathcal{V}^q, f_0 \in \mathcal{B}^q)$, $m \geqslant 0$, we can construct a hierarchy of the corresponding Lax operators

$$A_m = \sum_{i=0}^{m} V_i L^{m-i} \qquad (V_i \in \mathcal{V}_0^r,\ 0 \leqslant i \leqslant m) \qquad m \geqslant 0$$

according to the method of Ma (1991a). The space $\mathrm{Span}\{A_m | m \geqslant 0\}$ spanned by the hierarchy of Lax operators of that kind usually satisfies the above-mentioned conditions (1), (2), and thus it often forms a Lie subalgebra of $\mathcal{M}_\infty$. The cases of KdV, AKNS, dispersive long-wave and Boussinesq hierarchies have been discussed in Cheng and Li (1990), Li and Cheng (1991), and Zhang and Cheng (1990). Ma (1991c) has considered the cases of general hierarchies of integrable equations.

In the following, we discuss three kinds of special subalgebras of the Lax operator algebra $\langle \mathcal{M}_\infty, [\![\cdot,\cdot]\!]\rangle$ and the related problem of symmetries.

3.1. Abelian subalgebras

Abelian subalgebras just correspond to the algebras of K-symmetries of integrable equations. Obviously we have the following general result.

Theorem 3.2. If $R_0 \sqsubseteq \mathcal{M}_\infty$ is an Abelian subalgebra and if $\partial E(R_0)/\partial t = \{\partial Z/\partial t | Z \in E(R_0)\} = 0$, then every equation $u_t = K$ $(K \in E(R_0))$ possesses K-symmetries $S_K = E(R_0)$ and S_K is an Abelian subalgebra of $E(\mathcal{M}_\infty)$.

Suppose that a hierarchy of Lax operators $A_m \in \mathcal{M}_\infty$, $m \geqslant 0$, commute with each other and that A_m, $m \geqslant 0$, have the eigenvector fields $X_m \in E(\mathcal{M}_\infty)$, $m \geqslant 0$. By corollary 2.4, we know that a hierarchy of equations $u_t = X_m$, $m \geqslant 0$, possess commutative flows. Further, if $\partial X_m/\partial t = 0$, $m \geqslant 0$, then every equation $u_t = X_i$, $(i \geqslant 0)$ possesses a hierarchy of commutative K-symmetries $\{X_m\}_{m=0}^{\infty}$ by the above theorem.

In general, nonlinear integrable equations always possess a hierarchy of K-symmetries. But there also exist some integrable equations, for example the Boussinesq equation, which possess two hierarchies of K-symmetries. We do not know, however, whether the other integrable equations (say, KdV and AKNS equations, etc) possess two hierarchies of K-symmetries too, or even more.

3.2. τ-subalgebras

In this subsection, we consider types of subalgebras which generates τ-symmetries (see Ma 1990) of integrable equations.

Definition 3.1. Let M be an algebra, $*$ its multiplication operation and R_0, R_1 two subalgebras of M. If R_0 is Abelian and $R_0 * R_1$, $R_1 * R_0 \sqsubseteq R_0$, then $R = R_0 + R_1$ is called a τ-subalgebra of M. If R is also a Lie algebra, then R called a Lie τ-subalgebra of M.

Hereditary algebras introduced by Fuchssteiner (1990) are a special case of τ-algebras.

Theorem 3.3. Let $R_0, R_1 \sqsubseteq \mathcal{M}_\infty$ and $\partial E(R_i)/\partial t = \{\partial Z/\partial t | Z \in E(R_i)\} = 0$, $i = 0, 1$. If $R = R_0 + R_1$ is a τ-subalgebra of $\mathcal{M}_\infty$, then (1) every equation $u_t = K$ $(K \in E(R_0))$ possesses a set of K-symmetries $S_K = E(R_0)$ and a set of τ-symmetries $S_\tau = \{\tau_Y = t[K, Y] + Y | Y \in E(R_1)\}$; (2) $S = S_K + S_\tau$ is a Lie τ-subalgebra of $E(\mathcal{M}_\infty)$ and has the commutator relations

$$[X_1, X_2] = 0 \qquad X_1, X_2 \in E(R_0) \tag{3.1}$$

$$[X, \tau_Y] = [X, Y] \qquad X, \in E(R_0) \qquad Y, \in E(R_1) \tag{3.2}$$

$$[\tau_{Y_1}, \tau_{Y_2}] = \tau_{[Y_1, Y_2]} \qquad Y_1, Y_2 \in E(R_1)\,. \tag{3.3}$$

Proof. (1) Noticing that $E(R_0)$ is Abelian and that $\partial E(R_0)/\partial t = 0$, we find that S_K is a set of K-symmetries of $u_t = K$. In addition, we easily see that any vector field $Y \in E(R_1)$ is a K-generator of first order with characteristic 0 (for definition, see Ma 1991b) and thus we deduce that S_τ is a set of τ-symmetries of $u_t = K$.

(2) We only need to prove (3.2) and (3.3). For any $X \in E(R_0)$, $Y, Y_1, Y_2 \in E(R_1)$, we have

$$[X, \tau_Y] = [X, t[K, Y] + Y] = t[X, [K, Y]] + [X, Y] = [X, Y]$$

$$\begin{aligned}[\tau_{Y_1}, \tau_{Y_2}] &= [t[K, Y_1] + Y_1, t[K, Y_2] + Y_2] = t[[K, Y_1], Y_2] + t[Y_1, [K, Y_2]] + [Y_1, Y_2] \\ &= t[K, [Y_1, Y_2]] + [Y_1, Y_2] = \tau_{[Y_1, Y_2]}\,.\end{aligned}$$

Therefore (3.2) and (3.3) hold. The proof is completed. □

From this theorem, we know that $E(R_1)$ is a set of common master symmetries of first order of the equations $u_t = K,\ K \in E(R_0)$.

Generally for integrable equations, $R_0 = \mathrm{Span}\{A_m \in \mathcal{M}_\infty | m \geqslant 0\}$, $R_1 = \mathrm{Span}\{B_n \in \mathcal{M}_\infty | n \geqslant 0\}$. Let $A_m, B_n\,(m,n \geqslant 0)$ have the eigenvector fields $X_m, Y_n \in E(\mathcal{M}_\infty)\,(m,n \geqslant 0)$, respectively. This moment, $E(R_0) = \mathrm{Span}\{X_m \in E(\mathcal{M}_\infty) | m \geqslant 0\}$, $E(R_1) = \mathrm{Span}\{Y_n \in E(\mathcal{M}_\infty) | n \geqslant 0\}$. Thus we deduce from theorem 3.3 that every equation $u_t = X_i\,(i \geqslant 0)$ possesses a hierarchy of K-symmetries $\{X_m\}_{m=0}^\infty$ and a hierarchy of τ-symmetries $\{\tau_n^{(i)} = t[X_i, Y_n] + Y_n\}_{n=0}^\infty$ (at the same time, $\{Y_n\}_{n=0}^\infty$ is a hierarchy of master symmetries of first order), and $S = S_K + S_\tau = E(R_0) + \{t[X_i, Y] + Y | Y \in E(R_1)\}$ constitutes a Lie τ-subalgebra of $E(\mathcal{M}_\infty)$:

$$[X_m, X_n] = 0 \qquad m,n \geqslant 0 \tag{3.4}$$

$$[X_m, \tau_n^{(i)}] = [X_m, Y_n] \qquad m,n \geqslant 0 \tag{3.5}$$

$$[\tau_m^{(i)}, \tau_n^{(i)}] = t\,[X_i, [Y_m, Y_n]] + [Y_m, Y_n] \qquad m,n \geqslant 0\,. \tag{3.6}$$

These types of τ-algebras are often hereditary algebras (see Fuchssteiner 1990). Moreover these types of τ-algebras of many known integrable equations have been presented in Chen *et al* (1985, 1982, 1983, 1987); Cheng, Li and Bullough (1988); Cheng (1988, 1989, 1990, 1991); Ma (1990) and Li (1990) etc.

3.3. Master subalgebras

Definition 3.2. Let M be an algebra, $*$ its multiplication operation and $R_i \sqsubseteq M$, $i \geqslant 0$. If $R_i * R_j \sqsubseteq R_{i+j-1}$, $R_{-1} = 0$, $i,j \geqslant 0$, then $R = \sum_{i=0}^\infty R_i = \{\sum_{i=0}^n A_i | n \geqslant 0,\ A_i \in R_i,\ 0 \leqslant i \leqslant n\}$ is called a master subalgebra of M. If R is also a Lie algebra, then R called a Lie master subalgebra of M.

Here we have not required that R_i, $i \geqslant 0$, are subalgebras of M and that $R = \sum_{i=0}^\infty R_i$ is a direct sum. Therefore Lie master algebras are different from Z-graded Lie algebras (see Kac 1985) although there exist some similarities between the two. Obviously, master algebras are the extension of τ-algebras. When $R = \sum_{i=0}^\infty R_i$ is a master subalgebra, $R_0 + R_1$ is certain to be a τ-subalgebra.

Theorem 3.4. Let $R_i \sqsubseteq \mathcal{M}_\infty$, $i \geqslant 0$, and $\partial E(R_i)/\partial t = \{\partial Z/\partial t | Z \in E(R_i)\} = 0$, $i \geqslant 0$. If $R = \sum_{i=0}^\infty R_i$ is a master subalgebra of $\mathcal{M}_\infty$, then (1) every equation $u_t = K\ (K \in E(R_0))$ possesses a set of K-symmetries $S_K = E(R_0)$ and a set of time-polynomial-dependent symmetries

$$S_\sigma = \bigcup_{k=1}^\infty S_\sigma^{(k)} \qquad S_\sigma^{(k)} = \left\{ \sigma_Y^{(k)} = \sum_{i=0}^k \frac{t^i}{i!} \hat{K}^i Y \;\middle|\; Y \in E(R_k) \right\} \qquad k \geqslant 1$$

where $\hat{K}$ denotes its adjoint operator; (2) $S = S_K + \sum_{k=1}^\infty S_\sigma^{(k)}$ is a Lie master subalgebra of $E(\mathcal{M}_\infty)$ and possesses the commutator relations

$$[X_1, X_2] = 0 \qquad X_1, X_2 \in E(R_0) \tag{3.7}$$

$$[X, \sigma_Y^{(k)}] = \sigma_{[X,Y]}^{(k-1)} \qquad X, \in E(R_0) \qquad Y, \in E(R_k) \qquad k \geqslant 1 \tag{3.8}$$

$$[\sigma_{Y_1}^{(k)}, \sigma_{Y_2}^{(l)}] = \sigma_{[Y_1,Y_2]}^{(k+l-1)} \qquad Y_1 \in E(R_k) \qquad Y_2 \in E(R_l) \qquad k,l \geqslant 1\,. \tag{3.9}$$

Proof. (1) Obviously, S_K is a set of K-symmetries of $u_t = K$. In addition, we easily see that any vector field $Y \in E(R_k)$ $(k \geqslant 1)$ is a K-generator of order k with characteristic 0. Therefore $\sigma_Y^{(k)}$ is a time-polynomial-dependent symmetry of $u_t = K$.

(2) Noticing that

$$\underbrace{[E(R_0),[E(R_0),\ldots,[E(R_0)}_{k+1},E(R_k)]\ldots]] = 0 \qquad k \geqslant 0$$

we easily obtain (3.7), (3.8). In the following, we prove (3.9). For $Y_1 \in E(R_k), Y_2 \in E(R_l)$, $k,l \geqslant 1$, we have

$$[\sigma_{Y_1}^{(k)},\sigma_{Y_2}^{(l)}] = \sum_{i=0}^{k}\sum_{j=0}^{l}\frac{t^{i+j}}{i!j!}[\hat{K}^iY_1,\hat{K}^jY_2] = \sum_{r=0}^{k+l-1}\sum_{i+j=r}\frac{t^r}{i!j!}[\hat{K}^iY_1,\hat{K}^jY_2]$$

$$= \sum_{r=0}^{k+l-1}\frac{t^r}{r!}\sum_{i+j=r}\frac{r!}{i!j!}[\hat{K}^iY_1,\hat{K}^jY_2] = \sum_{r=0}^{k+l-1}\frac{t^r}{r!}\hat{K}^r[Y_1,Y_2] = \sigma_{[Y_1,Y_2]}^{(k+l-1)}$$

which implies that (3.9) holds.

By (3.7)–(3.9), we see that $S = S_K + \sum_{k=1}^{\infty} S_\sigma^{(k)}$ is a Lie master subalgebra of $E(\mathcal{M}_\infty)$. Now the proof is completed. □

From theorem 3.4, we can find that the equations $u_t = K$, $K \in E(R_0)$, have a set of common master symmetries of order k: $E(R_k)\,(k \geqslant 1)$.

Generally for integrable equations, $R_0 = \mathrm{Span}\{A_m \in \mathcal{M}_\infty | m \geqslant 0\}$, $R_i = \mathrm{Span}\{B_{in} \in \mathcal{M}_\infty | n \geqslant 0\}$, $i \geqslant 1$. Let A_m, B_{in} $(i \geqslant 1,\ m,n \geqslant 0)$ have the eigenvector fields $X_m, Y_{in} \in E(\mathcal{M}_\infty)$ $(i \geqslant 1,\ m,n \geqslant 0)$, respectively. Then $E(R_0) = \mathrm{Span}\{X_m \in E(\mathcal{M}_\infty) | m \geqslant 0\}$, $E(R_i) = \mathrm{Span}\{Y_{in} \in E(\mathcal{M}_\infty) | n \geqslant 0\}$, $i \geqslant 1$. Thus every equation $u_t = X_i$ $(i \geqslant 0)$ possesses a hierarchy of K-symmetries $\{X_m\}_{m=0}^{\infty}$ and infinitely many hierarchies of time-polynomial-dependent symmetries $\{\sigma_{Y_{kn}}^{(k)} = \sum_{j=0}^{k}\frac{t^j}{j!}\hat{X}_i^jY_{kn}\}_{n=0}^{\infty}$, $k \geqslant 1$ (at the same time, $\{Y_{kn}\}_{n=0}^{\infty}$ $(k \geqslant 1)$ is a hierarchy of master symmetries of order k), and

$$S = S_K + \sum_{k=1}^{\infty} S_\sigma^{(k)} = E(R_0) + \sum_{k=1}^{\infty}\left\{\sigma_Y^{(k)} = \sum_{j=0}^{k}\frac{t^j}{j!}\hat{X}_i^jY \;\middle|\; Y \in E(R_k)\right\}$$

constitutes a Lie master subalgebra of $E(\mathcal{M}_\infty)$:

$$[X_m, X_n] = 0 \qquad m,n \geqslant 0 \tag{3.10}$$

$$[X_m, \sigma_{Y_{ln}}^{(l)}] = \sigma_{[X_m,Y_{ln}]}^{(l-1)} \qquad l \geqslant 1 \quad m,n \geqslant 0 \tag{3.11}$$

$$[\sigma_{Y_{km}}^{(k)}, \sigma_{Y_{ln}}^{(l)}] = \sigma_{[Y_{km},Y_{ln}]}^{(k+l-1)} \qquad k,l \geqslant 1 \quad m,n \geqslant 0\,. \tag{3.12}$$

Finally, we point out that τ-algebras and master algebras of the Lax operator algebra play an analogous role to recursion operators in discussing the algebraic properties of integrable equations. Furthermore, the integrable equations in $1+1$ dimensions usually have τ-algebras and those in $2+1$ dimensions often have master algebras, which is a remarkable difference between these kinds of integrable equations.

4. Applications to integrable equations

In this section we want to present some applications of τ-subalgebras and master subalgebras to integrable equations. We shall mainly construct a τ-subalgebra and a master algebra and thus derive a hierarchy of K-symmetries and infinitely many hierarchies of time-polynomial-dependent symmetries for the well known KP hierarchy of integrable equations. Certainly, the theory of τ-subalgebras and master subalgebras may also applied to other hierarchies of integrable equations, for example, MKP and Caudrey–Dodd–Gibbon–Katera–Sawada hierarchies, etc.

We choose the following $(2+1)$-dimensional spectral operator L:

$$L = \alpha\partial_y + \partial_x^2 + u \qquad \alpha \in C \qquad \alpha \neq 0 \qquad u = u(x,y,t) \in S(R^2, R)\,. \quad (4.1)$$

Evidently, the Gateaux derivative operator reads as $L'[X] = X$, $X \in \mathcal{B}$, and thus is injective. Let

$$B = \sum_{k,l=0}^{m} b_{kl}\partial_x^k\partial_y^l \qquad b_{kl} \in \mathcal{B} \qquad 0 \leqslant k, l \leqslant m \qquad m \geqslant 0$$

be a Lax operator. Noticing that $\partial_y = \frac{1}{\alpha}(L - \partial_x^2 - u)$, we can rewrite B as $B = \sum_{i=0}^{n} A_i L^i$, where A_i, $0 \leqslant i \leqslant n$, are polynomials only in ∂_x. In this way, we have

$$[B, L] = \sum_{i=0}^{n}[A_i, L]L^i.$$

Because $[B, L]$ is a multiplication operator, we can further obtain $[B, L] = [A_0, L]$ by comparing the degrees of ∂_y. Therefore we may only consider the following differential polynomial operator in ∂_x:

$$A = \sum_{k=0}^{m} a_k\partial_x^k \qquad a_k \in \mathcal{B} \qquad 0 \leqslant k \leqslant m \qquad m \geqslant 0 \qquad (4.2)$$

as a candidate for Lax operators. We can find by direct computation that the differential operator A with the form (4.2) is a Lax operator, i.e. there exists a vector field $X \in \mathcal{B}$ such that $[A, L] = L'[X] = X$ if and only if a_k, $0 \leqslant k \leqslant m$, satisfies the following equations

$$\begin{cases} a_{mx} = 0 \\ \alpha a_{my} + 2a_{m-1,x} = 0 \\ \alpha a_{m-1,y} + 2a_{m-2,x} - a_m u_x = 0 \\ \alpha a_{ky} + a_{kxx} + 2a_{k-1,x} - \displaystyle\sum_{i=k+1}^{m} \binom{i}{k} a_i \partial_x^{i-k} u = 0 \qquad 1 \leqslant k \leqslant m-2 \end{cases} \qquad (4.3)$$

and

$$X = \sum_{k=1}^{m} a_k\partial_x^k u - \alpha\partial_y a_0 - \partial_x^2 a_0\,. \qquad (4.4)$$

Set the space W as

$$W = \left\{ f + g \middle| f = \sum_{i,j \geqslant 0} c_{ij} x^i y^j ,\ c_{ij} \in C, \right.$$
$$\left. g = \sum\nolimits_{k \in Z}^{i,j,l \geqslant 0} c_{ijkl} x^i y^j \partial_x^k \partial_y^l u,\ c_{ijkl} \in C \right\} \tag{4.5}$$

in which $\partial_x^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) \mathrm{d}x'$. We introduce the inverse operator ∂_x^{-1} of ∂_x over the space W as follows:

$$\partial_x^{-1} h = \partial_x^{-1}(f+g) = \int_0^x f \,\mathrm{d}x' + \tfrac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) g \,\mathrm{d}x' \qquad h = f + g \in W\,. \tag{4.6}$$

Furthermore suppose that $C[y]$ denotes all polynomials in y, and $C_n[y]$ $(n \geqslant 0)$, all polynomials in y with degrees less than $n+1$. We write

$$a_k = a_{k1} + a_{k2} \qquad a_{k1} = a_k|_{u=0} \qquad a_{k2} = a_k - a_{k1} \qquad 0 \leqslant k \leqslant m\,. \tag{4.7}$$

We choose the following group of coefficients a_k, $0 \leqslant k \leqslant m$, satisfying (4.3):

$$\begin{cases} a_{m1} = c_m \qquad a_{m2} = 0 \\ a_{m-1,1} = -\dfrac{\alpha}{2}\partial_x^{-1}\partial_y a_{m1} + c_{m-1} \qquad a_{m-1,2} = 0 \\ a_{k-1,1} = \left(-\dfrac{\alpha}{2}\partial_x^{-1}\partial_y - \dfrac{1}{2}\partial_x\right) a_{k1} + c_{k-1} \\ a_{k-1,2} = \left(-\dfrac{\alpha}{2}\partial_x^{-1}\partial_y - \dfrac{1}{2}\partial_x\right) a_{k2} + \dfrac{1}{2}\partial_x^{-1} \displaystyle\sum_{i=k+1}^{m} \binom{i}{k} a_i \partial_x^{i-k} u \\ \qquad 1 \leqslant k \leqslant m-1 \end{cases} \tag{4.8}$$

where $c_k \in C[y]$, $0 \leqslant k \leqslant m$. It is easy to show that $a_k = a_{k1} + a_{k2} \in W$, $0 \leqslant k \leqslant m$. For every group of $c_k \in C[y]$, $0 \leqslant k \leqslant m$, according to (4.8) we can uniquely determine a Lax operator $A = \sum_{k=0}^{m} a_k \partial_x^k$, more precisely $A = P(c_0, \ldots, c_m)$. Set

$$\mathcal{N}_\infty = \{A = P(c_0, \ldots, c_m) | \, m \geqslant 0 \qquad c_k \in C[y] \qquad 0 \leqslant k \leqslant m\} \tag{4.9}$$

$$R_i = \{A = P(c_0, \ldots, c_m) | \, m \geqslant 0 \qquad c_k \in C_i[y] \qquad 0 \leqslant k \leqslant m\} \qquad i \geqslant 0\,. \tag{4.10}$$

By (4.8), we easily obtain the following two basic results.

Proposition 4.1. Let $A = P(c_0, \ldots, c_m) = \sum_{k=0}^{m} a_k \partial_x^k \in \mathcal{N}_\infty$. Then we have (1) A is t-function multi-linear with respect to $c_0, \ldots, c_m$; (2) $A|_{u=0} = \sum_{k=0}^{m} a_{k1}\partial_x^k$; (3) if $A|_{u=0} = 0$, then $A = 0$.

Proposition 4.2. Let $A = P(c_0, \ldots, c_m) = \sum_{k=0}^{m} a_k \partial_x^k \in R_i$ $(m, i \geqslant 0)$. If we set

$$\bar{a}_k = (a_k - c_k)|_{u=0} = a_{k1} - c_k \qquad 0 \leqslant k \leqslant m \tag{4.11}$$

then when $i = 0$, $\bar{a}_k = 0$, $0 \leqslant k \leqslant m$; and when $i \geqslant 1$, $\bar{a}_m = 0$ and $\bar{a}_k$, $0 \leqslant k \leqslant m-1$, are polynomials in x, y with degrees less than i with respect to y.

Now we begin to verify that $\mathcal{N}_\infty = \sum_{i=0}^{\infty} R_i$ is a master algebra. To this end, we first give two lemmas.

Lemma 4.1. When $A \in R_0$, $[A, L]|_{u=0} = X|_{u=0} = 0$; and when $A \in R_i\,(i \geqslant 1)$, $[A, L]|_{u=0} = X|_{u=0}$ is a polynomial in x, y with degrees less than i with respect to y.

Proof. Suppose that

$$A = P(c_0, \dots, c_m) = \sum_{k=0}^{m} a_k \partial_x^k .$$

By (4.4), we have

$$\begin{aligned}[A, L]|_{u=0} = X|_{u=0} &= (-\alpha\partial_y a_0 - \partial_x^2 a_0)|_{u=0} = -\alpha\partial_y a_{01} - \partial_x^2 a_{01} \\ &= -\alpha\partial_y(\bar{a}_0 + c_0) - \partial_x^2 \bar{a}_0\end{aligned}$$

where $\bar{a}_0 = (a_0 - c_0)|_{u=0}$. Thus the desired result follows from proposition 4.2, which completes the proof. □

Lemma 4.2. When $A, B \in R_0$, $[A, B]|_{u=0} = 0$; and when $A \in R_i$, $B \in R_j$ $(i, j \geqslant 0,\ i + j \geqslant 1)$, the coefficients of the differential operator $[A, B]|_{u=0}$ are polynomials in x, y with degrees less than $i + j$ with respect to y.

Proof. Assume that

$$A = P(c_0, \cdots, c_m) = \sum_{k=0}^{m} a_k \partial_x^k \qquad B = P(d_0, \cdots, d_n) = \sum_{l=0}^{n} b_l \partial_x^l .$$

Then we have

$$A|_{u=0} = \sum_{k=0}^{m} (\bar{a}_k + c_k)\partial_x^k = \sum_{k=0}^{m-1} \bar{a}_k \partial_x^k + \sum_{k=0}^{m} c_k \partial_x^k$$

$$B|_{u=0} = \sum_{l=0}^{n} (\bar{b}_l + d_l)\partial_x^l = \sum_{l=0}^{n-1} \bar{b}_l \partial_x^l + \sum_{l=0}^{n} d_l \partial_x^l$$

where $\bar{a}_k = (a_k - c_k)|_{u=0}$, $0 \leqslant k \leqslant m$, $\bar{b}_l = (b_l - d_l)|_{u=0}$, $0 \leqslant l \leqslant n$. Hence, we obtain

$$\begin{aligned}[A, B]|_{u=0} &= [A|_{u=0}, B|_{u=0}] = \left[\sum_{k=0}^{m-1} \bar{a}_k \partial_x^k + \sum_{k=0}^{m} c_k \partial_x^k, \sum_{l=0}^{n-1} \bar{b}_l \partial_x^l + \sum_{l=0}^{n} d_l \partial_x^l\right] \\ &= \left[\sum_{k=0}^{m-1} \bar{a}_k \partial_x^k, \sum_{l=0}^{n-1} \bar{b}_l \partial_x^l\right] + \left[\sum_{k=0}^{m-1} \bar{a}_k \partial_x^k, \sum_{l=0}^{n} d_l \partial_x^l\right] + \left[\sum_{k=0}^{m} c_k \partial_x^k, \sum_{l=0}^{n-1} \bar{b}_l \partial_x^l\right].\end{aligned}$$

From this, we obtain by proposition 4.2 the desired result. □

Theorem 4.1. Let $\mathcal{N}_\infty$, R_i, $i \geqslant 0$, be determined, by (4.9), (4.10), respectively. Then $\langle \mathcal{N}_\infty = \sum_{i=0}^{\infty} R_i, [\![\cdot,\cdot]\!]\rangle$ forms a master algebra and thus $\langle E(\mathcal{N}_\infty) = \sum_{i=0}^{\infty} E(R_i), [\cdot,\cdot]\rangle$ forms a Lie master algebra.

Proof. We only need to prove that

$$[\![R_i, R_j]\!] \sqsubseteq R_{i+j-1}, R_{-1} = 0 \qquad i, j \geqslant 0. \tag{4.12}$$

Let $A \in R_i$, $B \in R_j$ $(i, j \geqslant 0)$ and $X \in E(R_i)$, $Y \in E(R_j)$ be the eigenvector fields of A, B, respectively. Then we have

$$\begin{aligned}[\![A, B]\!]|_{u=0} &= (A'[Y] - B'[X] + [A, B])|_{u=0} \\ &= A'[Y|_{u=0}]|_{u=0} - B'[X|_{u=0}]|_{u=0} + [A, B]|_{u=0}\,.\end{aligned} \tag{4.13}$$

When $i + j = 0$, i.e. $i = j = 0$, it follows from (4.13) and lemmas 4.1, 4.2 that $[\![A, B]\!]|_{u=0} = 0$. Thus by the result (3) of proposition 4.1, we obtain $[\![A, B]\!] = 0$, i.e. $[\![A, B]\!] \in R_{-1}$. When $i + j \geqslant 1$, it follows similarly from (4.13) and lemmas 4.1, 4.2 that the coefficients of the differential operator $[\![A, B]\!]|_{u=0}$ are polynomials in x, y with degrees less than $i + j$ with respect to y. Thus by the result (2) of proposition 4.1 and proposition 4.2, we obtain $[\![A, B]\!] \in R_{i+j-1}$. Summing up, we see that the relation (4.12) holds, which is the desired result. □

We choose

$$A_m = P(\underbrace{0, \ldots, 0}_{m}, \tfrac{1}{3}(6\alpha)^{m-1}) = \sum_{k=0}^{m} a_k^{(m)} \partial_x^k \qquad m \geqslant 0 \tag{4.14}$$

$$B_{in} = P(\underbrace{0, \ldots, 0}_{n}, \tfrac{1}{3}(6\alpha)^{n-1} y^i) = \sum_{l=0}^{n} b_l^{(in)} \partial_x^l \qquad i \geqslant 1 \qquad n \geqslant 0\,. \tag{4.15}$$

Then the corresponding eigenvector fields read as

$$X_m = [A_m, L] = \sum_{k=1}^{m} a_k^{(m)} \partial_x^k u - \alpha \partial_y a_0^{(m)} - \partial_x^2 a_0^{(m)} \qquad m \geqslant 0 \tag{4.16}$$

$$Y_{in} = [B_{in}, L] = \sum_{l=1}^{n} b_l^{(in)} \partial_x^l u - \alpha \partial_y b_0^{(in)} - \partial_x^2 b_0^{(in)} \qquad i \geqslant 1 \qquad n \geqslant 0\,. \tag{4.17}$$

By the first result of proposition 4.1

$$R_0 = \mathrm{Span}\{A_m | m \geqslant 0\} \qquad R_i = \mathrm{Span}\{B_{jn} | n \geqslant 0 \quad 0 \leqslant j \leqslant i\} \qquad i \geqslant 1 \tag{4.18}$$

$$E(R_0) = \mathrm{Span}\{X_m | m \geqslant 0\} \qquad E(R_i) = \mathrm{Span}\{Y_{jn} | n \geqslant 0 \quad 0 \leqslant j \leqslant i\} \qquad i \geqslant 1\,. \tag{4.19}$$

According to (4.8), (4.16) we can work out

$$A_0 = \frac{1}{18\alpha} \qquad X_0 = 0 \qquad A_1 = \tfrac{1}{3}\partial_x \qquad X_1 = \tfrac{1}{3}u_x$$
$$A_2 = 2\alpha u + 2\alpha\partial_x^2 \qquad X_2 = -2\alpha^2 u_y$$

$$
\begin{aligned}
A_3 &= -9\alpha^3\partial_x^{-1}u_y + 9\alpha^2 u_x + 18\alpha^2 u\partial_x + 12\alpha^2\partial_x^3 \\
X_3 &= 18\alpha^2 u u_x + 3\alpha^2 u_{xxx} + 9\alpha^4\partial_x^{-1}u_{yy} \\
A_4 &= 36\alpha^5\partial_x^{-2}u_{yy} - 36\alpha^4 u_y + 72\alpha^3 u^2 + 72\alpha^3 u_{xx} \\
&\qquad - (72\alpha^4\partial_x^{-1}u_y - 144\alpha^3 u_x)\partial_x + 144\alpha^3 u\partial_x^2 + 72\alpha^3\partial_x^4 \\
X_4 &= -72\alpha^4 u_x\partial_x^{-1}u_y - 36\alpha^6\partial_x^{-2}u_{yyy} - 144\alpha^4 u u_y - 36\alpha^4 u_{xxy}\,.
\end{aligned}
$$

The hierarchy of equations $u_t = X_m,\ m \geqslant 0$, is referred to as the KP hierarchy of equations. By theorem 4.1, this KP hierarchy has a τ-algebra $\langle \mathcal{L}_\infty = R_0 + R_1, [\![\cdot,\cdot]\!]\rangle$ (or $\langle E(\mathcal{L}\infty) = E(R_0) + E(R_1), [\cdot,\cdot]\rangle$) and a master algebra $\langle \mathcal{N}_\infty = \sum_{i=0}^\infty R_i, [\![\cdot,\cdot]\!]\rangle$ (or $\langle E(\mathcal{N}_\infty) = \sum_{i=0}^\infty E(R_i), [\cdot,\cdot\rangle$). When $\alpha = i/\sqrt{3}$, the KP hierarchy of equations $u_t = X_m,\ m \geqslant 0$, is just the normal KP hierarchy of equations in the literature. In particular, the equation $u_t = X_3$ is just the normal KP equation $u_t = \partial_x^{-1}u_{yy} - u_{xxx} - 6uu_x$.

Based on Theorem 4.1, we obtain at once the following consequence.

Theorem 4.2. Every KP equation $u_t = X_i\ (i \geqslant 0)$, given by (4.16), possesses a hierarchy of common K-symmetries $\{X_m\}_{m=0}^\infty$ and infinitely many hierarchies of time-polynomial-dependent symmetries

$$
\left\{ \sigma_{Y_{kn}}^{(k)} = \sum_{j=0}^{k} \frac{t^j}{j!}\hat{X}_i^j Y_{kn} \right\}_{n=0}^{\infty} \qquad k \geqslant 1\,.
$$

With (4.8) and (4.17), we can similarly calculate the first four master symmetries of first order and the corresponding Lax operators:

$$
\begin{aligned}
&B_{10} = yA_0 \qquad Y_{10} = -\tfrac{1}{18} \\
&B_{11} = yA_1 - 3\alpha^2 x A_0 \qquad Y_{11} = yX_1 \\
&B_{12} = yA_2 - 3\alpha^2 x A_1 + \tfrac{1}{2}\alpha^2 \qquad Y_{12} = yX_2 - 3\alpha^2 x X_1 - 2\alpha^2 u \\
&B_{13} = yA_3 - 3\alpha^2 x A_2 + 3\alpha^3\partial_x - 3\alpha^3\partial_x^{-1}u \\
&Y_{13} = yX_3 - 3\alpha^2 x X_2 + 9\alpha^3 u_x + 12\alpha^4\partial_x^{-1}u_y
\end{aligned}
$$

and the first three master symmetries of order $k\,(\geqslant 2)$ and their corresponding Lax operators:

$$
\begin{aligned}
&B_{k0} = y^k A_0 \qquad Y_{k0} = -\frac{k}{18}y^{k-1} \\
&B_{k1} = y^k A_1 - 3k\alpha^2 x y^{k-1}A_0 \qquad Y_{k1} = y^k X_1 + \tfrac{1}{6}k(k-1)\alpha^2 x y^{k-2} \\
&B_{k2} = y^k A_2 - 3k\alpha^2 x y^{k-1}A_1 + \tfrac{1}{2}k\alpha^2 y^{k-1} + \tfrac{1}{4}k(k-1)\alpha^3 x^2 y^{k-2} \\
&Y_{k2} = y^k X_2 - 3k\alpha^2 x y^{k-1}X_1 - 2k\alpha^2 y^{k-1}u \\
&\qquad\quad - k(k-1)\alpha^3 y^{k-2} - \tfrac{1}{4}k(k-1)(k-2)\alpha^4 x^2 y^{k-3}
\end{aligned}
$$

where we accept $\mathrm{O}(y^{-1}) = 0$.

By considering the ys' degrees of coefficients of master symmetries, we can find that the master symmetries of order $k\,(\geqslant 1)$ proposed in many references, for example Case and Monge (1989), Cheng (1990), Fuchssteiner (1983), Gu and Li (1990), all belong to the kth space $E(R_k)$ of master symmetries, i.e. they are all linear combinations of the master symmetries $Y_{kn},\ 0 \leqslant i \leqslant k,\ n \geqslant 0$, given by (4.17).

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